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Asymptotic expansion of solutions to the drift-diffusion equation with large initial data

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ABSTRACT

We consider the large-time behavior of the solution to the initial value problem for the Nernst–Planck type drift-diffusion equation in whole spaces. In the L^p-framework, the global existence and the decay of the solution were shown. Moreover, the second-order asymptotic expansion of the solution as $t \to \infty$ was derived. We also deduce the higher-order asymptotic expansion of the solution. Especially, we discuss the contrast between the odd-dimensional case and the even-dimensional case.

1. Introduction

We consider the large-time behavior of the solution to the Cauchy problem for the following drift-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ -\Delta \psi = -\kappa u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geqslant 3$, $\kappa = \pm 1$, $\partial_t := \partial/\partial_t$, $\Delta := \sum_{j=1}^n \partial_j^2$ and $\partial_j := \partial/\partial x_j$. The drift-diffusion equation with $\kappa = 1$ is known as the model of the plasma dynamics. When $\kappa = -1$, the drift-diffusion equation stands for the motion of the gravitational interaction particles in astronomy (cf. [3]). The drift-diffusion equation is first considered as a Neumann problem in a bounded domain with $\kappa = 1$ (see [19]). In this case, the time global existence of the solution was shown by Fang and Ito [6]. Moreover, Biler and Dolbeault [1] and Jüngel [11] proved the asymptotic stability of the solution. For the Cauchy problem (1), the time local well-posedness was given by Kurokiba and Ogawa [18]. In particular, in the case $\kappa = 1$, the time global existence of the solution with a large initial data was discussed. Upon the suitable condition (for instance $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $u_0 \geqslant 0$, and $\kappa = 1$), it is known that the solution to (1) exists globally in time and satisfies $\|u(t)\|_p \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}$ for $1 \leqslant p \leqslant \infty$ (see [1,14,25,26] and Proposition 6 in Section 2). In this paper, we shall derive the asymptotic expansion of the solution to (1) as $t \to \infty$ in $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$. Especially, by considering the three-dimensional case and the four-dimensional case, we state the difference between the odd-dimensional cases and the even-dimensional cases. Before stating our conclusions, we refer to some preceding studies for other equations. There are similar equations appearing in other phenomena. For instance, the Navier–Stokes equation describes an incompressible fluid flow (cf. [8,13]). The Keller–Segel equation appearing in the model of the chemotaxis (for example, we refer to [10,15,16,20,21,24,27,29]). The Keller–Segel equation is known as a very much related equation to the drift-diffusion equation with $\kappa = -1$ (cf. [17]). For those equations, the asymptotic profile of the solution is considered by many mathematicians. This was first considered by

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Escobedo and Zuazua [5] for the solution to the convection diffusion equation. For the linear heat equation with a general potential, the large-time behavior of the solution was obtained (see [9] and references therein). For some semilinear heat equations with a fractional dissipation, the large-time behavior of the solution was derived (for example, see [2]). For the solution to the Navier–Stokes equation, Carpio [4] showed the asymptotic expansion of the solution in two-dimensional case. In multi-dimensional cases, the asymptotic expansion of the solution was given by Fujigaki and Miyakawa [7]. The large-time behavior of the solution to the Keller–Segel equation in $L^p(\mathbb{R}^n)$ was considered by Nagai, Syukuinn and Umesako [22]. They showed the first-order asymptotic expansion of the solution to the Keller–Segel equation with a small initial data. Moreover, the higher-order asymptotic expansion of the solution to the following parabolic–parabolic type Keller–Segel equation was given by Kato [12] and Nagai and Yamada [23]:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \partial_t \psi - \Delta \psi + \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $n \geqslant 1$. It was shown that the asymptotic expansion of the solution to (2) contains the correction term. We should remark that the correction term of this type never occurs in the case of the Navier–Stokes equation. Moreover, when n is even, Yamada [30,31] showed that the asymptotic expansion of the solution to (2) contains the logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n+1}{2}}$. On the other hand, if n is odd, the asymptotic expansion of the solution has no logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k}{2}}$ for $0 \leqslant k \leqslant n+1$.

In order to analyze the Cauchy problem (1), we introduce the following integral equation:

$$u(t) = e^{t\Delta} u_0 + \kappa \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u \nabla (-\Delta)^{-1} u\big)(s)\, ds, \tag{3}$$

where the semigroup $\{e^{t\Delta}\}_{t \geqslant 0}$ is provided by $e^{t\Delta}\varphi := G(t) * \varphi$, $G(t,x) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$, and the operator $(-\Delta)^{-1}$ is represented as $(-\Delta)^{-1}\varphi(x) := \frac{1}{n(2-n)\omega_n} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}}\, dy$ for $\varphi \in L^p(\mathbb{R}^n)$, $1 < p < n/2$, where ω_n denotes the surface area of the n-dimensional unit sphere. The solution to (3) is called the *mild solution* of (1). Kurokiba and Ogawa [18] showed that the mild solution solves the Cauchy problem (1). Hereafter, we analyze the mild solution to obtain the asymptotic expansion of the solution. Set the following symbols: $M := \int_{\mathbb{R}^n} u_0(y)\, dy$, $m = (m_1, \dots, m_n) := -\int_{\mathbb{R}^n} y u_0(y)\, dy$. Then, we introduce the functions $V_0 = V_0(t,x)$ and $V_1 = V_1(t,x)$ as

$$V_0(t,x) := MG(t,x), \qquad V_1(t,x) := m \cdot \nabla G(t,x). \tag{4}$$

Moreover, when $n = 3$, we define the function $J = J(t,x)$ by

$$J(t,x) := \kappa \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0 \nabla (-\Delta)^{-1} V_0\big)(s)\, ds. \tag{5}$$

We note that, for any $\lambda > 0$, the following relations hold:

$$\begin{cases} V_0(t,x) = \lambda^n V_0\big(\lambda^2 t, \lambda x\big), \quad V_1(t,x) = \lambda^{n+1} V_1\big(\lambda^2 t, \lambda x\big) & \text{for } n \geqslant 3, \\ J(t,x) = \lambda^4 J\big(\lambda^2 t, \lambda x\big) & \text{for } n = 3. \end{cases} \tag{6}$$

Furthermore, the property $J \not\equiv 0$ holds if $M \neq 0$ (for the details, see Proposition 9 in Section 2). For the function V_0 and the solution u to (1), we can conclude the following estimate under the suitable condition (cf. [14,25]): $\|u(t) - V_0(1+t)\|_p = o(t^{-\frac{n}{2}(1-\frac{1}{p})})$ as $t \to \infty$ for $1 \leqslant p \leqslant \infty$. In particular, in the case $n = 3$, the following estimate holds (for the details, see [25] and Proposition 10 in Section 3): $\|u(t) - V_0(1+t) - V_1(1+t) - J(1+t)\|_p = o(t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}})$ as $t \to \infty$ for $1 \leqslant p \leqslant \infty$. In this argument, it seems to be natural to give V_1 as

$$\begin{aligned} V_1(t,x) = -\nabla G(t,x) \cdot \Bigg(& \int_{\mathbb{R}^3} y u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^3} V_0 \nabla (-\Delta)^{-1} V_0(1+s,y)\, dy\, ds \\ & + \int_0^\infty \int_{\mathbb{R}^3} \big(u \nabla (-\Delta)^{-1} u(s,y) - V_0 \nabla (-\Delta)^{-1} V_0(1+s,y)\big)\, dy\, ds \Bigg). \end{aligned} \tag{7}$$

Indeed, for the Keller–Segel equation (2) with $n = 1$, the second-order asymptotic expansion of the solution is given by the similar form (cf. [12,23]). For our equation, by employing the integration by parts, we can conclude that the integrands

$u\nabla(-\Delta)^{-1}u$ and $V_0\nabla(-\Delta)^{-1}V_0$ are vanishing. Thus, the definition (4) is appropriate to give the second-order asymptotic expansion of the solution.

We shall find out the higher-order asymptotic expansion of the solution when $n=3$. For this aim, we introduce the functions as follows:

$$K(t,x) := -\frac{\kappa}{3}\Delta G(t,x)\int_0^t\int_{\mathbb{R}^3} y\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds \tag{8}$$

and

$$\begin{aligned} V_2(t,x) :=& \sum_{2l+|\beta|=2}\frac{(-1)^l\partial_t^l\nabla^\beta G(t,x)}{\beta!}\int_{\mathbb{R}^3}(-y)^\beta u_0(y)\,dy \\ &- \kappa\sum_{|\beta|=1}\nabla^\beta\nabla G(t,x)\cdot\int_0^\infty\int_{\mathbb{R}^3} y^\beta\big\{u\nabla(-\Delta)^{-1}u(s,y) - V_0\nabla(-\Delta)^{-1}V_0(1+s,y) \\ &- \big(V_0\nabla(-\Delta)^{-1}(V_1+J) + (V_1+J)\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\big\}\,dy\,ds \\ &- \frac{\kappa}{3}\Delta G(t,x)\int_0^\infty\int_{\mathbb{R}^3} y\cdot\big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds \\ &+ \kappa\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds \\ &+ \kappa\int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(t,x)(-y)^\beta\Big) \\ &\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds. \end{aligned} \tag{9}$$

The function V_2 satisfies the following scaling property:

$$V_2(t,x) = \lambda^5 V_2\big(\lambda^2 t,\lambda x\big)\quad \text{for any } \lambda>0. \tag{10}$$

Moreover, the function K satisfies

$$\begin{aligned} K(t,x) &= -\frac{\kappa}{3}\Delta G(t,x)\int_0^t (1+s)^{-1}\,ds\int_{\mathbb{R}^3}(1+s)^{-1/2}y \\ &\quad\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)\big(1,(1+s)^{-1/2}y\big)(1+s)^{-3/2}\,dy \\ &= -\frac{\kappa}{3}\log(1+t)\Delta G(t,x)\int_{\mathbb{R}^3}\eta\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)(1,\eta)\,d\eta, \end{aligned}$$

where we use the scaling property (6) in the first equality, and put $\eta := (1+s)^{-1/2}y$ in the second equality. Furthermore, we can expect $K\not\equiv 0$ since the integrand $\eta\cdot(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0)(1,\eta)$ is even. We give the large-time behavior of the solution for three-dimensional case in the following theorem.

Theorem 1. *Assume that $n=3$, $u_0\in L_2^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Let $u_0\geqslant 0$ in the case $\kappa=1$. On the other hand, in the case $\kappa=-1$, assume that $\|u_0\|_{L^1\cap L^\infty}$ is sufficiently small. Let u be the solution to* (1)*, and the functions V_0, V_1, J, K, V_2 be defined by* (4), (5), (8), (9)*. Then, the following estimate holds*:

$$\big\|u(t) - V_0(1+t) - V_1(1+t) - J(1+t) - K(1+t) - V_2(1+t)\big\|_p = o\big(t^{-\gamma-1}\big)\quad \text{as } t\to\infty,$$

where $1\leqslant p\leqslant\infty$, $\gamma := \frac{3}{2}(1-\frac{1}{p})$. And if $M\neq 0$, then the terms J and K satisfy $J, K\not\equiv 0$.

For a small initial data, the same estimate as in Theorem 1 was already given (cf. [32]). Actually, we can derive this estimate for a large initial data.

When $n=4$, the following estimate holds under the suitable condition (see [25]): $\|u(t)-V_0(1+t)-V_1(1+t)\|_p = o(t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}})$ as $t\to\infty$ for $1\leqslant p\leqslant\infty$. In order to obtain the higher-order asymptotic expansion of the solution, we introduce the following functions:

$$\begin{aligned}
\tilde V_2(t,x) &:= \sum_{2l+|\beta|=2} \frac{(-1)^l \partial_t^l \nabla^\beta G(t,x)}{\beta!} \int_{\mathbb{R}^4} (-y)^\beta u_0(y)\,dy \\
&\quad + \kappa \sum_{|\beta|=1} \nabla^\beta \nabla G(t,x) \cdot \int_0^\infty \int_{\mathbb{R}^4} (-y)^\beta \big(u\nabla(-\Delta)^{-1}u(s,y) - V_0\nabla(-\Delta)^{-1}V_0(1+s,y)\big)\,dy\,ds \\
&\quad + \kappa \int_0^t \int_{\mathbb{R}^4} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(t,x)(-y)^\beta\Big) \cdot \big(V_0\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds, \\
\tilde K(t,x) &:= -\frac{\kappa}{4}\Delta G(t,x) \int_0^t \int_{\mathbb{R}^4} y\cdot \big(V_0\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds. \qquad (11)
\end{aligned}$$

For the function $\tilde V_2$ and any $\lambda>0$, the relation $\tilde V_2(t,x)=\lambda^6 \tilde V_2(\lambda^2 t,\lambda x)$ is satisfied. Furthermore, by the relation (6), the following equality holds:

$$\begin{aligned}
\tilde K(t,x) &= -\frac{1}{4}\Delta G(t,x) \int_0^t (1+s)^{-1} \int_{\mathbb{R}^4} (1+s)^{-1/2} y\cdot \big(V_0\nabla(-\Delta)^{-1}V_0\big)\big(1,(1+s)^{-1/2}y\big)(1+s)^{-2}\,dy\,ds \\
&= -\frac{1}{4}\log(1+t)\Delta G(t,x) \int_{\mathbb{R}^4} \eta\cdot \big(V_0\nabla(-\Delta)^{-1}V_0\big)(1,\eta)\,d\eta,
\end{aligned}$$

where we put $\eta:=(1+s)^{-1/2}y$ in the second equality. We can expect that $\tilde K \not\equiv 0$ since the integrand $\eta\cdot(V_0\nabla(-\Delta)^{-1}V_0)(1,\eta)$ is even. The solution to (1) satisfies the following estimate when $n=4$.

Theorem 2. *Let $n=4$ and $u_0\in L^1_2(\mathbb{R}^4)\cap L^\infty(\mathbb{R}^4)$. In addition, let $u_0\geqslant 0$ in the case $\kappa=1$. Assume that $\|u_0\|_{L^1\cap L^\infty}$ is sufficiently small when $\kappa=-1$. Let u be the solution to (1) and the functions $V_0, V_1, \tilde K, \tilde V_2$ be defined by (4), (11). Then, the solution u satisfies*

$$\big\|u(t)-V_0(1+t)-V_1(1+t)-\tilde K(1+t)-\tilde V_2(1+t)\big\|_p = o\big(t^{-\gamma-1}\big) \quad \text{as } t\to\infty,$$

for $1\leqslant p\leqslant\infty$, $\gamma:=2(1-\frac{1}{p})$. Moreover, the function $\tilde K$ satisfies $\tilde K\not\equiv 0$ if $\int_{\mathbb{R}^4} u_0(y)\,dy\neq 0$.

Theorems 1 and 2 state that the asymptotic profile of the solution contains the logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-1}$ when $n=3$ or 4. When $n=3$, it seems natural that a logarithmic term appears in the asymptotic profile of the solution at the rate $t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}$. Indeed, in the second-order asymptotic expansion provided by the formal form (7), the term $\int_0^t\int_{\mathbb{R}^3} V_0\nabla(-\Delta)^{-1}V_0(1+s,y)\,dy\,ds$ seems to have the logarithmic order. Namely,

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} V_0\nabla(-\Delta)^{-1}V_0(1+s,y)\,dy\,ds &= \int_0^t (1+s)^{-1} \int_{\mathbb{R}^3} V_0\nabla(-\Delta)^{-1}V_0\big(1,(1+s)^{-1/2}y\big)(1+s)^{-3/2}\,dy\,ds \\
&= \log(1+t)\int_{\mathbb{R}^3} V_0\nabla(-\Delta)^{-1}V_0(1,\eta)\,d\eta,
\end{aligned}$$

where we use the relation (6) in the first equality, and we put $\eta:=(1+s)^{-1/2}y$ in the second equality. Actually, this term is vanishing since the integrand $V_0\nabla(-\Delta)^{-1}V_0(1,\eta)$ is odd. Hence, the asymptotic expansion of the solution has no logarithmic term at the rate $t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}$.

We should remark that the functions V_2 and $\tilde V_2$ in Theorems 1 and 2 contain some extra terms. For example, in the second term on the right-hand side of (4), the integrands $y_j V_0 \partial_k(-\Delta)^{-1}V_0$ are vanishing if $j\neq k$. For simplicity, we leave those terms.

Notation. In this paper, we use the following notation. For $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, we denote the inner product by $x\cdot y := x_1y_1+\cdots+x_ny_n$. The Fourier transform is defined by $\hat\varphi(\xi)=[\varphi]^\wedge(\xi)=\mathcal{F}[\varphi](\xi):=$

$(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$, where $i := \sqrt{-1}$. The partial derivative operators are denoted by $\partial_t := \partial/\partial_t$, $\partial_j := \partial/\partial x_j$ ($1 \leqslant j \leqslant n$). Moreover, we denote $\nabla := (\partial_1, \dots, \partial_n)$ and $\Delta := \sum_{j=1}^n \partial_j^2$. For $x \in \mathbb{R}^n$, let $\langle x\rangle := \sqrt{1+|x|^2}$. We denote L^p and $W^{s,p}$ the Lebesgue spaces and the Sobolev spaces for $1 \leqslant p \leqslant \infty$ and $s \in \mathbb{R}$. The norm of $L^p(\mathbb{R}^n)$ is represented by $\|\cdot\|_{L^p}$ or $\|\cdot\|_p$, and the norm of $\varphi \in W^{s,p}(\mathbb{R}^n)$ is $\|\varphi\|_{W^{s,p}} := \|\varphi\|_p + \|(-\Delta)^{s/2}\varphi\|_p$. For an exponent $1 \leqslant p \leqslant \infty$, Hölder's dual exponent p' is provided by $\frac{1}{p'} = 1 - \frac{1}{p}$. Let $L^p_m(\mathbb{R}^n)$ be the weighted L^p space with $\|\varphi\|_{L^p_m} := \|\langle x\rangle^m\varphi\|_p$. In $L^p(\mathbb{R}^n)$, the decay rate of the fundamental solution is denoted by $\gamma := \frac{n}{2}(1-\frac{1}{p})$. The set of nonnegative integers is represented by $\mathbb{Z}_+$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, the α-th order derivative is denoted by $\nabla^\alpha := \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$. Moreover, we let $\alpha! := \alpha_1!\cdots\alpha_n!$. For a vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, we denote $a^\alpha := a_1^{\alpha_1}\cdots a_n^{\alpha_n}$. The length of a multi-index is given by $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Various constants are simply denoted by C.

2. Preliminaries

In this section, we prepare several lemmas and propositions in order to prove our main theorems. The following lemma gives an estimate for $\nabla\psi$ in (1) and some fractional integrals.

Lemma 3 *(Hardy–Littlewood–Sobolev's inequality). Let $n \geqslant 2$, $1 < p < n$ and $\frac{n}{n-1} < p_* < \infty$ with $\frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}$. Then, there exists a constant $C > 0$ such that $\|\nabla(-\Delta)^{-1} f\|_{p_*} \leqslant C\|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$.*

Proof. For the proof of Lemma 3, see [33, 2.8.4. Theorem]. We omit the details. □

The following lemma is well known for the estimates of the heat semi-group.

Lemma 4. *Let $n \geqslant 1$, $\alpha, \beta \in \mathbb{Z}_+^n$, and $u_0 \in L^1_{|\alpha|}(\mathbb{R}^n)$. Then, it holds that $x^\alpha\nabla^\beta e^{t\Delta}u_0 \in L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$ and $t > 0$. Moreover, there exists a positive constant $C > 0$ such that $\|x^\alpha\nabla^\beta e^{t\Delta}u_0\|_p \leqslant Ct^{-\gamma-\frac{|\beta|-|\alpha|}{2}}\|u_0\|_{L^1_{|\alpha|}}$ for all $t > 0$, where $\gamma := \frac{n}{2}(1-\frac{1}{p})$.*

Proof. Hausdorff–Young's inequality immediately gives Lemma 4. We omit the proof. □

The scaling properties (6) give the following proposition.

Proposition 5. *Let $n = 3$ or 4, the functions V_0 and V_1 be defined by (4). When $n = 3$, let the function J be given by (5). Suppose that $1 \leqslant p \leqslant \infty$, and $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leqslant 1$. Then, the conditions $\|x^\beta V_0(1)\|_p, \|x^\beta V_1(1)\|_p < \infty$ and $\|x^\beta J(1)\|_p < \infty$ are satisfied. Moreover, the following equalities hold: $\|x^\beta V_k(1+t)\|_p = (1+t)^{-\gamma-\frac{k}{2}+\frac{|\beta|}{2}}\|x^\beta V_k(1)\|_p$ ($k = 0, 1$) and $\|x^\beta J(1+t)\|_p = (1+t)^{-\gamma-\frac{1}{2}+\frac{|\beta|}{2}}\|x^\beta J(1)\|_p$ for any $t > 0$, where $\gamma := \frac{n}{2}(1-\frac{1}{p})$.*

Proof. We should confirm that $\|x^\beta V_0(1)\|_p, \|x^\beta V_1(1)\|_p, \|x^\beta J(1)\|_p < \infty$ for $|\beta| \leqslant 1$ and $1 \leqslant p \leqslant \infty$. The definitions (4) immediately give $\|x^\beta V_0(1)\|_p < \infty$ and $\|x^\beta V_1(1)\|_p < \infty$. We check $\|x^\beta J(1)\|_p < \infty$ when $n = 3$. The function $|x^\beta J(1)|$ satisfies

$$\begin{aligned}\left|x^\beta J(1)\right| \leqslant{}& \int_0^1\int_{\mathbb{R}^3}\left|(x-y)^\beta\nabla G(1-s, x-y)\right|\left|V_0\nabla(-\Delta)^{-1}V_0(s,y)\right|dy\,ds \\ &+ \int_0^1\int_{\mathbb{R}^3}\left|\nabla G(1-s, x-y)\right|\left|y^\beta V_0\nabla(-\Delta)^{-1}V_0(s,y)\right|dy\,ds\end{aligned}$$

since the simple relation $x^\beta = (x-y)^\beta + y^\beta$ holds. A combination of this and Hausdorff–Young's inequality gives $\|x^\beta J(1)\|_p < \infty$ for $1 \leqslant p \leqslant \infty$. Hence, we obtain $\|x^\beta V_0(1)\|_p, \|x^\beta V_1(1)\|_p < \infty$ and $\|x^\beta J(1)\|_p < \infty$ for any $1 \leqslant p \leqslant \infty$. By those arguments together with the relations (6), we conclude the proof. □

We give the underlying arguments for the solution by the following proposition.

Proposition 6. *Let $n \geqslant 2$, $\kappa = \pm 1$, and $n/2 < p < n$. Then, for any $u_0 \in L^p(\mathbb{R}^n)$, there exist a positive constant T and a unique solution u to (3) such that*

$$u \in C\big([0,T); L^p(\mathbb{R}^n)\big) \cap C\big((0,T); W^{2,p}(\mathbb{R}^n)\big) \cap C^1\big((0,T); L^p(\mathbb{R}^n)\big).$$

Moreover, if the initial data satisfies $u_0 \geqslant 0$, then $u(t) \geqslant 0$ holds. If $u_0 \in L^1(\mathbb{R}^n)$, then the relation $\int_{\mathbb{R}^n} u(t,x)dx = \int_{\mathbb{R}^n} u_0(x)\,dx$ holds for any $t > 0$. Furthermore, assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $u_0 \geqslant 0$ when $\kappa = 1$, and let $\|u_0\|_{L^1\cap L^\infty}$ be sufficiently small when $\kappa = -1$. Then, the solution u to (1) exists globally in time and satisfies

$$\big\|u(t)\big\|_p \leqslant C(1+t)^{-\gamma} \quad \text{for any } t > 0 \text{ and } 1 \leqslant p \leqslant \infty,\ \gamma := \frac{n}{2}\left(1 - \frac{1}{p}\right). \tag{12}$$

Proof. For the proof of Proposition 6, see [14,18,26]. Here, we omit the details. □

In addition, the solution to (1) satisfies the following estimate.

Proposition 7. *Let $n \geqslant 2$ and $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume that $u_0 \geqslant 0$ when $\kappa = 1$. Let $\|u_0\|_{L^1\cap L^\infty}$ be sufficiently small when $\kappa = -1$. Then, the solution u to (1) satisfies*

$$\big\|xu(t)\big\|_p \leqslant C(1+t)^{-\gamma+\frac{1}{2}} \quad \text{for any } t > 0 \text{ and } 1 \leqslant p < \frac{n}{n-1},\ \gamma := \frac{n}{2}\left(1 - \frac{1}{p}\right). \tag{13}$$

Proof. Proposition 7 can be proved by the same way as in the proof of [31, Proposition 3.1]. Now, we prove this by another way. Since the solution is represented by (3), we have

$$x_j u(t) = x_j e^{t\Delta} u_0 + x_j \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u\nabla(-\Delta)^{-1}u\big)(s)\,ds \tag{14}$$

for $1 \leqslant j \leqslant n$. By Hausdorff–Young's inequality, the following inequality holds:

$$\begin{aligned}
\big\|x_j e^{t\Delta}u_0\big\|_p &\leqslant \left\| \int_{\mathbb{R}^n} (x_j - y_j) G(t, x-y) u_0(y)\,dy \right\|_p + \left\| \int_{\mathbb{R}^n} G(t, x-y) y_j u_0(y)\,dy \right\|_p \\
&\leqslant C\big\|x_j G(t)\big\|_1 \|u_0\|_p + C\big\|G(t)\big\|_1 \|y_j u_0\|_p \\
&\leqslant C(1+t)^{1/2}\big(\|u_0\|_{L^p} + \big\||y|^2 u_0\big\|_1^{1/2} \|u_0\|_{p/(2-p)}^{1/2}\big),
\end{aligned}$$

where we use the relation $\|y_j u_0\|_p \leqslant \||y|^2 u_0\|_1^{1/2} \|u_0\|_{p/(2-p)}^{1/2}$ in the third inequality. Hence, $\|x_j e^{t\Delta} u_0\|_p$ is bounded on $0 \leqslant t \leqslant 1$. Thus, applying Lemma 4, we obtain

$$\big\|x_j e^{t\Delta}u_0\big\|_p \leqslant C(1+t)^{-\gamma+\frac{1}{2}} \quad \text{for any } t > 0 \text{ and } 1 \leqslant p < n/(n-1),\ \gamma := \frac{n}{2}\left(1 - \frac{1}{p}\right). \tag{15}$$

On the other hand, we can split the nonlinear part on the right-hand side of (14) into

$$\begin{aligned}
x_j \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u\nabla(-\Delta)^{-1}u\big)(s)\,ds &= \int_0^t \int_{\mathbb{R}^n} (x_j - y_j) \nabla G(t-s, x-y) \cdot \big(u\nabla(-\Delta)^{-1}u(s,y)\big)\,dy\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \nabla G(t-s, x-y) \big(y_j u \nabla(-\Delta)^{-1}u(s,y)\big)\,dy\,ds.
\end{aligned} \tag{16}$$

First, we consider the case $1 < p < n/(n-1)$. Employing Hausdorff–Young's inequality, Lemma 3, and the decay estimate (12) for (16), it holds that

$$\begin{aligned}
&\left\| x_j \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u\nabla(-\Delta)^{-1}u\big)(s)\,ds \right\|_p \\
&\quad \leqslant \int_0^t \big(\big\|x_j \nabla G(t-s)\big\|_p \big\|u\nabla(-\Delta)^{-1}u(s)\big\|_1 + \big\|\nabla G(t-s)\big\|_p \big\|y_j u(s)\big\|_p \big\|\nabla(-\Delta)^{-1}u(s)\big\|_{p'} \big)\,ds \\
&\quad \leqslant C\int_0^t (t-s)^{-\gamma}(1+s)^{-\frac{n}{2}+\frac{1}{2}}\,ds + C\int_0^t (t-s)^{-\gamma-\frac{1}{2}}(1+s)^{-\frac{n}{2}+1}\big((1+s)^{\gamma-\frac{1}{2}}\big\|y_j u(s)\big\|_p\big)\,ds.
\end{aligned} \tag{17}$$

Hence, applying (15) and (17) to (14), we obtain

$$(1+t)^{\gamma-\frac{1}{2}}\left\|x_j u(t)\right\|_p \leqslant C + C(1+t)^{\gamma-\frac{1}{2}}\int_0^t (t-s)^{-\gamma-\frac{1}{2}}(1+s)^{-\frac{n}{2}+1}\left((1+s)^{\gamma-\frac{1}{2}}\left\|y_j u(s)\right\|_p\right)ds.$$

Employing the Gronwall lemma, we conclude that $(1+t)^{\gamma-\frac{1}{2}}\|x_j u(t)\|_p$ does not blow up in finite time. Moreover, we obtain

$$(1+t)^{\gamma-\frac{1}{2}}\left\|x_j u(t)\right\|_p \leqslant C + C(1+t)^{-\frac{n}{2}+1}\sup_{0<\tau<t}\left((1+\tau)^{\gamma-\frac{1}{2}}\left\|y_j u(\tau)\right\|_p\right)$$

for any $1<p<n/(n-1)$, $\gamma := \frac{n}{2}(1-\frac{1}{p})$. Since $(1+t)^{\gamma-\frac{1}{2}}\|x_j u(t)\|_p$ does not blow up in finite time, we conclude the desired estimate (13) for any $1<p<n/(n-1)$. Next, we show (13) with $p=1$ by applying the case $1<p<n/(n-1)$. Let $1<\rho<n/(n-1)$, then by the relation (16) and Hausdorff–Young's inequality, we have

$$\begin{aligned}&\left\|x_j\int_0^t \nabla e^{(t-s)\Delta}\cdot\left(u\nabla(-\Delta)^{-1}u\right)(s)\,ds\right\|_1\\ &\quad\leqslant \int_0^t\left\{\left\|x_j\nabla G(t-s)\right\|_1\left\|u(s)\right\|_\rho\left\|\nabla(-\Delta)^{-1}u(s)\right\|_{\rho'}+\left\|\nabla G(t-s)\right\|_1\left\|y_j u(s)\right\|_\rho\left\|\nabla(-\Delta)^{-1}u(s)\right\|_{\rho'}\right\}ds.\end{aligned}$$

Thus, applying Lemmas 3 and 4, the decay estimate (12), and the estimate (13) with $p:=\rho$, we obtain

$$\left\|x_j\int_0^t \nabla e^{(t-s)\Delta}\cdot\left(u\nabla(-\Delta)^{-1}u\right)(s)\,ds\right\|_1 \leqslant C\int_0^t\left\{(1+s)^{-\frac{n}{2}+\frac{1}{2}}+(t-s)^{-1/2}(1+s)^{-\frac{n}{2}+1}\right\}ds\leqslant C. \tag{18}$$

Applying (15) and (18) to (14), we obtain the desired estimate (13) with $p=1$. Thus, we conclude the proof. □

In order to prove the estimates for the higher-order asymptotic expansions of the solution, we prepare the estimates for the first-order asymptotic expansion of the solution.

Proposition 8. *Let $n\geqslant 3$ and $u_0\in L^1_1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Assume that $u_0\geqslant 0$ if $\kappa=1$. Let $\|u_0\|_{L^1\cap L^\infty}$ be sufficiently small when $\kappa=-1$. Let V_0 be defined by* (4)*. Then, the solution u to* (1) *satisfies*

$$\left\|u(t)-V_0(1+t)\right\|_p\leqslant C(1+t)^{-\gamma-\frac{1}{2}}\quad \text{for } 1\leqslant p\leqslant\infty, \tag{19}$$

where $\gamma:=\frac{n}{2}(1-\frac{1}{p})$. Moreover, the following estimates hold:

$$\begin{cases}\left\|u\nabla(-\Delta)^{-1}u(t)-V_0\nabla(-\Delta)^{-1}V_0(1+t)\right\|_p\leqslant C(1+t)^{-\gamma-\frac{n}{2}} & \text{for } 1\leqslant p<\infty,\\ \left\|x\left\{u\nabla(-\Delta)^{-1}u(t)-V_0\nabla(-\Delta)^{-1}V_0(1+t)\right\}\right\|_p\leqslant C(1+t)^{-\gamma-\frac{n-1}{2}} & \text{for } 1\leqslant p<\dfrac{n}{n-1},\end{cases} \tag{20}$$

where $\gamma:=\frac{n}{2}(1-\frac{1}{p})$.

Proof. For the proof of the estimate (19), see [14,25]. The estimates (20) are given by a combination of the inequality (12), Lemma 3, Propositions 5, 7, and the estimate (19). □

Before closing this section, we confirm that the functions J, K and $\tilde{K}$ satisfy $J, K, \tilde{K}\not\equiv 0$. Here, the functions J, K and $\tilde{K}$ are defined by (5), (8) and (9), respectively.

Proposition 9. *Let $n=3$, the functions J, K be defined by* (5) *and* (8)*. Then, the functions J, K satisfy $J, K\not\equiv 0$ if $M:=\int_{\mathbb{R}^n}u_0(y)\,dy\neq 0$. When $n=4$, let $M\neq 0$ and $\tilde{K}$ be defined by* (11)*. Then the function $\tilde{K}$ satisfies $\tilde{K}\not\equiv 0$.*

Proof. In order to prove $J\not\equiv 0$, we show $J(1,0)\neq 0$. Using the Parseval equality, we see that

$$J(1,0)=\kappa M^2\int_0^1\int_{\mathbb{R}^3}\nabla G(1-s,-y)\cdot\left(G\nabla(-\Delta)^{-1}G\right)(s,y)\,dy\,ds$$

$$= -\kappa M^2 \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta\cdot\xi}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta\,d\xi\,ds$$

$$= -\kappa M^2 \sum_{j=1}^3 \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta_j\xi_j}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta\,d\xi\,ds,$$

since $[G(s)]^\wedge(\xi) = e^{-s|\xi|^2}$. Hence, applying Fubini's theorem and the relation

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta_j\xi_j}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta\,d\xi = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta\,d\xi \quad (1 \leqslant j \leqslant 3),$$

we obtain

$$J(1,0) = -3\kappa M^2 \int_0^1 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\iint_{\mathbb{R}\times\mathbb{R}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1 \right) d\xi'\,d\eta'\,ds, \tag{21}$$

where we denote that $\xi' := (\xi_2, \xi_3)$ and $\eta' := (\eta_2, \eta_3)$. We split the integrand on the right-hand side of (21) into the positive part and the negative part:

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1$$

$$= \left(\left(\iint_{\{\xi_1,\eta_1\geqslant 0\}} + \iint_{\{\xi_1,\eta_1<0\}} \right) + \left(\iint_{\{\xi_1\geqslant 0,\,\eta_1<0\}} + \iint_{\{\xi_1<0,\,\eta_1\geqslant 0\}} \right) \right) \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1$$

$$= 2\left(\iint_{\{\xi_1,\eta_1\geqslant 0\}} + \iint_{\{\xi_1\geqslant 0,\,\eta_1<0\}} \right) \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1, \tag{22}$$

where we use the following relation in the second equality:

$$\left(\iint_{\{\xi_1,\eta_1<0\}} + \iint_{\{\xi_1<0,\,\eta_1\geqslant 0\}} \right) \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1$$

$$= \left(\iint_{\{\xi_1,\eta_1\geqslant 0\}} + \iint_{\{\xi_1\geqslant 0,\,\eta_1<0\}} \right) \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1.$$

By the change of variables, we see that the negative part on the right-hand side of (22) is represented by

$$\iint_{\{\xi_1\geqslant 0,\,\eta_1<0\}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1$$

$$= -\iint_{\{\xi_1,\eta_1\geqslant 0\}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta|^2} e^{-s(|\eta_1+\xi_1|^2+|\eta'-\xi'|^2)}\,d\eta_1\,d\xi_1, \tag{23}$$

where we denote that $\xi' := (\xi_2, \xi_3)$ and $\eta' := (\eta_2, \eta_3)$. Substituting (23) into (22), we have

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta-\xi|^2-s|\eta|^2}\,d\eta_1\,d\xi_1$$

$$= 3 \iint_{\{\xi_1,\eta_1\geqslant 0\}} \frac{\eta_1\xi_1}{|\eta|^2} e^{-(1-s)|\xi|^2-s|\eta|^2} \left(e^{-s|\eta-\xi|^2} - e^{-s(|\eta_1+\xi_1|^2+|\eta'-\xi'|^2)} \right) d\eta_1\,d\xi_1 > 0,$$

where we use the relation $e^{-s|\eta-\xi|^2} > e^{-s(|\eta_1+\xi_1|^2+|\eta'-\xi'|^2)}$ for any $s > 0$ and $(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $\xi_1\eta_1 \neq 0$. Namely, the integrand on the right-hand side of (21) is positive for any $s \in (0,1)$ and $(\xi',\eta') \in \mathbb{R}^2 \times \mathbb{R}^2$. Thus, we conclude $J(1,0) \neq 0$. Next, we prove that $K \not\equiv 0$. For this purpose, we show $C_K := \int_{\mathbb{R}^3} y \cdot (G\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}G)(1,y)\,dy \neq 0$ if

$M := \int_{\mathbb{R}^3} u_0(y)\,dy \neq 0$. We have already seen that the constant C_K gives $K(t,x) = \frac{\kappa M C_K}{3}\log(1+t)\Delta G(t,x)$. By the Parseval equality, the constant C_K is represented by

$$\begin{aligned} C_K &= \int_{\mathbb{R}^3} \big(\big[yG(1)\big]^{\wedge}\cdot\big[\nabla(-\Delta)^{-1}J(1)\big]^{\wedge} + \big[J(1)\big]^{\wedge}\big[y\cdot\nabla(-\Delta)^{-1}G(1)\big]^{\wedge}\big)\,d\xi \\ &= \int_{\mathbb{R}^3} \left(i\nabla\big(e^{-|\xi|^2}\big)\cdot\frac{i\xi}{|\xi|^2} + i\nabla\cdot\left(\frac{i\xi}{|\xi|^2}e^{-|\xi|^2}\right)\right)\big[J(1)\big]^{\wedge}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^3} \left(4 - \frac{1}{|\xi|^2}\right)e^{-|\xi|^2}\big[J(1)\big]^{\wedge}(\xi)\,d\xi. \end{aligned} \tag{24}$$

Now, since the odd integrand $\frac{\eta}{|\eta|^2}e^{-|\eta|^2}$ is vanishing, we have that

$$\begin{aligned} \big[J(1)\big]^{\wedge}(\xi) &= \kappa M^2\int_0^1 \big[\nabla G(1-s)\big]^{\wedge}(\xi)\cdot\int_{\mathbb{R}^3}\big[G(s)\big]^{\wedge}(\eta-\xi)\big[\nabla(-\Delta)^{-1}G(s)\big]^{\wedge}(\eta)\,d\eta\,ds \\ &= \kappa M^2\int_0^1 (i\xi)e^{-(1-s)|\xi|^2}\cdot\int_{\mathbb{R}^3} e^{-s|\eta-\xi|^2}\frac{i\eta}{|\eta|^2}e^{-s|\eta|^2}\,d\eta\,ds \\ &= -\kappa M^2\int_0^1\int_{\mathbb{R}^3}\frac{\xi\cdot\eta}{|\eta|^2}e^{-|\xi|^2-2s|\eta|^2+2s\xi\cdot\eta}\,d\eta\,ds \\ &= -\kappa M^2\int_0^1\int_{\mathbb{R}^3}\frac{\xi\cdot\eta}{|\eta|^2}e^{-|\xi|^2-2s|\eta|^2}\big(e^{2s\xi\cdot\eta}-1\big)\,d\eta\,ds. \end{aligned} \tag{25}$$

Substituting (25) into (24), and employing the mean-valued theorem, we obtain that

$$\begin{aligned} C_K &= -4\kappa M^2\int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\xi\cdot\eta}{|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2+2s\xi\cdot\eta}\,d\eta\,d\xi\,ds \\ &\quad + \kappa M^2\int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\xi\cdot\eta}{|\xi|^2|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2}\big(e^{2s\xi\cdot\eta}-1\big)\,d\eta\,d\xi\,ds \\ &= -4\kappa M^2\int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\xi\cdot\eta}{|\eta|^2}e^{-2|\xi-\frac{s}{2}\eta|^2-s(2-\frac{s}{2})|\eta|^2}\,d\eta\,d\xi\,ds \\ &\quad + 2\kappa M^2\int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^s\frac{(\xi\cdot\eta)^2}{|\xi|^2|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2+2\sigma\xi\cdot\eta}\,d\sigma\,d\eta\,d\xi\,ds. \end{aligned} \tag{26}$$

We now calculate the first term on the right-hand side of (26). We put $\zeta := \xi - \frac{s}{2}\eta$, since the odd integrand $\zeta e^{-2|\zeta|^2}$ is vanishing, we have that

$$\begin{aligned} \int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\xi\cdot\eta}{|\eta|^2}e^{-2|\xi-\frac{s}{2}\eta|^2-s(2-\frac{s}{2})|\eta|^2}\,d\eta\,d\xi\,ds &= \int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(\zeta+\frac{s}{2}\eta)\cdot\eta}{|\eta|^2}e^{-2|\zeta|^2-s(2-\frac{s}{2})|\eta|^2}\,d\eta\,d\zeta\,ds \\ &= \frac{1}{2}\int_0^1\iint_{\mathbb{R}^3\times\mathbb{R}^3} se^{-2|\zeta|^2-s(2-\frac{s}{2})|\eta|^2}\,d\eta\,d\zeta\,ds. \end{aligned} \tag{27}$$

On the other hand, we derive the upper bound for the second term on the right-hand side of (26). Since the relations $|(\xi\cdot\eta)| < |\xi||\eta|$ and $-2|\xi|^2 - 2s|\eta|^2 + 2\sigma\xi\cdot\eta = -2|\xi - \frac{\sigma}{2}\eta|^2 - (2s - \frac{\sigma^2}{2})|\eta|^2$ hold for almost all $(\xi,\eta)\in\mathbb{R}^3\times\mathbb{R}^3$, we have

$$\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s \frac{(\xi\cdot\eta)^2}{|\xi|^2|\eta|^2} e^{-2|\xi|^2-2s|\eta|^2+2\sigma\xi\cdot\eta}\,d\sigma\,d\eta\,d\xi\,ds < \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s e^{-2|\xi-\frac{\sigma}{2}\eta|^2-(2s-\frac{\sigma^2}{2})|\eta|^2}\,d\sigma\,d\eta\,d\xi\,ds$$
$$= \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s e^{-2|\zeta|^2-(2s-\frac{\sigma^2}{2})|\eta|^2}\,d\sigma\,d\eta\,d\zeta\,ds, \tag{28}$$

where we put $\zeta := \xi - \frac{\sigma}{2}\eta$ and employ Fubini's theorem in the equality. Substituting (27) and (28) into (26), we see that $C_K \neq 0$. Thus, we conclude that $K \not\equiv 0$. At last, we prove that $\tilde{K} \not\equiv 0$. We have already seen that the function $\tilde{K}$ is represented by

$$\tilde{K}(t,x) = -\frac{\kappa M^2 C_{\tilde{K}}}{4}\log(1+t)\Delta G(t,x), \qquad C_{\tilde{K}} := \int_{\mathbb{R}^4} y\cdot\big(G\nabla(-\Delta)^{-1}G\big)(1,y)\,dy\,ds.$$

We can confirm that $C_{\tilde{K}} \neq 0$. Indeed, by the Parseval equality, the following equality holds:

$$C_{\tilde{K}} = \int_{\mathbb{R}^4} \big[yG(1)\big]^{\wedge}(\xi)\cdot\big[\nabla(-\Delta)^{-1}G(1)\big]^{\wedge}(\xi)\,d\xi$$
$$= \int_{\mathbb{R}^4} \big(-2i\xi e^{-|\xi|^2}\big)\cdot\left(\frac{i\xi}{|\xi|^2}e^{-|\xi|^2}\right)d\xi = 2\int_{\mathbb{R}^4} e^{-2|\xi|^2}\,d\xi \neq 0.$$

Thus, we conclude the proof. □

3. Proof of Theorem 1

Before proving Theorem 1, we prepare the auxiliary estimates.

Proposition 10. *Under the same assumption as in Theorem* 1, *the solution u to* (1) *satisfies the following estimate for any $1 \leqslant p < \infty$, $\gamma := \frac{3}{2}(1-\frac{1}{p})$*:

$$\big\|u(t) - V_0(1+t) - V_1(1+t) - J(1+t)\big\|_p \leqslant C(1+t)^{-\gamma-1}\log(2+t). \tag{29}$$

For $3/2 < p < \infty$, the following estimate is satisfied:

$$\big\|\nabla(-\Delta)^{-1}u(t) - \nabla(-\Delta)^{-1}(V_0+V_1+J)(1+t)\big\|_p \leqslant C(1+t)^{-\gamma-\frac{1}{2}}\log(2+t). \tag{30}$$

Moreover, the following estimate holds for $1 \leqslant p < \infty$:

$$\big\|u\nabla(-\Delta)^{-1}u(t) - \mathcal{V}(1+t)\big\|_p \leqslant C(1+t)^{-\gamma-2}\log(2+t), \tag{31}$$

where the $\mathbb{R}^3$-valued function $\mathcal{V} = \mathcal{V}(t,x)$ is defined by

$$\mathcal{V}(t,x) := \big(V_0\nabla(-\Delta)^{-1}V_0 + V_0\nabla(-\Delta)^{-1}(V_1+J) + (V_1+J)\nabla(-\Delta)^{-1}V_0\big)(t,x). \tag{32}$$

Proof. We prove the estimate (29). We split the solution u into the following parts:

$$u(t) = V_0(1+t) + V_1(1+t) + J(1+t) + r_1(t) + r_2(t) + r_3(t),$$

where

$$r_1(t,x) := e^{t\Delta}u_0 - V_0(1+t,x) - V_1(1+t,x),$$
$$r_2(t,x) := \int_0^t \nabla e^{(t-s)\Delta}u\nabla(-\Delta)^{-1}u(s)\,ds - J(t,x)$$
$$= \int_0^t\int_{\mathbb{R}^3} \big(\nabla G(t-s,x-y) - \nabla G(1+t,x)\big)\cdot\big(u\nabla(-\Delta)^{-1}u - V_0\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds,$$
$$r_3(t,x) := J(t,x) - J(1+t,x).$$

Here, we have used the relation $\int_{\mathbb{R}^3}(u\nabla(-\Delta)^{-1}u - V_0\nabla(-\Delta)^{-1}V_0)\,dy = 0$ in the transformation of r_2. It is well known that r_1 satisfies $\|r_1(t)\|_p \leqslant C(1+t)^{-\gamma-1}$. By the mean-valued theorem, the function r_2 can be represented as

$$r_2(t,x) = \int_0^{t/2}\int_{\mathbb{R}^3}\int_0^1 \partial_t\nabla G(t-\sigma s, x-y)(-s)\cdot\big(u\nabla(-\Delta)^{-1}u(s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,d\sigma\,dy\,ds$$
$$+\sum_{|\beta|=1}\int_0^{t/2}\int_{\mathbb{R}^3}\int_0^1 \nabla^\beta\nabla G(t, x-\sigma y)(-y)^\beta\cdot\big(u\nabla(-\Delta)^{-1}u(s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,d\sigma\,dy\,ds$$
$$+\int_{t/2}^{t}\int_{\mathbb{R}^3}\big(\nabla G(t-s,x-y) - \nabla G(t,x)\big)\cdot\big(u\nabla(-\Delta)^{-1}u(s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds.$$

Applying Proposition 8, we have

$$\big\|r_2(t)\big\|_p \leqslant C\int_0^{t/2}\big((t-s)^{-\gamma-\frac{3}{2}}s^{-\frac{3}{2}+1} + (t-s)^{-\gamma-1}(1+s)^{-1}\big)\,ds + C\int_{t/2}^{t}(t-s)^{-\frac{1}{2}}s^{-\gamma-\frac{3}{2}}\,ds$$
$$\leqslant Ct^{-\gamma-1}\log(1+t).$$

By the mean-valued theorem, we see that r_3 satisfies $\|r_3(t)\|_p = o(t^{-\gamma-1})$. Thus, we obtain the estimate (29). The estimates (30) and (31) can be proved by using the estimate (29) together with the Hölder inequality, Lemma 3, Propositions 5 and 8. Thus, we conclude the proof. $\square$

Proposition 10 gives the following estimate.

Proposition 11. *Under the same assumption as in Theorem* 1, *the following estimate holds for* $1 \leqslant p < 3/2$ *and* $1 \leqslant j \leqslant 3$:

$$\big\|x_j\big\{u\nabla(-\Delta)^{-1}u(t) - \mathcal{V}(1+t)\big\}\big\|_p \leqslant C(1+t)^{-\gamma-\frac{3}{2}}\log(1+t),$$

where the function $\mathcal{V} = \mathcal{V}(t,x)$ *is defined by* (32).

In order to prove Proposition 11, we prepare the Hörmander–Mikhlin type multiplier theorem obtained by Shibata and Shimizu [28].

Lemma 12. *(Cf. [28, Theorem 2.3].) Let* $n \in \mathbb{N}$, $\lambda > -n$ *and set* $\lambda = k + \mu - n$, *where* k *is a nonnegative integer and* $0 < \mu \leqslant 1$. *Let* $\varphi \in C^\infty(\mathbb{R}^n\backslash\{0\})$ *satisfy*

$$\begin{cases} \nabla^\alpha\varphi \in L^1(\mathbb{R}^n) & \text{for all } \alpha \in \mathbb{Z}_+^n \text{ with } |\alpha| \leqslant k, \\ \big|\nabla^\beta\varphi(\xi)\big| \leqslant C_\beta|\xi|^{\lambda-|\beta|} & \text{for all } \beta \in \mathbb{Z}_+^n \text{ with } |\beta| \leqslant k+1,\ \xi \neq 0. \end{cases}$$

Then, the following inequality holds: $\sup_{x\neq 0}|x|^{n+\lambda}|\mathcal{F}^{-1}[\varphi](x)| < \infty$.

Proof of Proposition 11. Since the definition (32), we can split $x_j(u\nabla(-\Delta)^{-1}u - \mathcal{V})$ into

$$x_j\big(u\nabla(-\Delta)^{-1}u - \mathcal{V}\big) = x_ju\nabla(-\Delta)^{-1}(u - V_0 - V_1 - J) + (u - V_0 - V_1 - J)x_j\nabla(-\Delta)^{-1}V_0$$
$$+ (u - V_0)x_j\nabla(-\Delta)^{-1}(V_1 + J). \tag{33}$$

We set $1 < r < 3/2$ and $3/2 < q < \infty$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. Then, by Propositions 7 and 10, the first term on the right-hand side of (33) satisfies

$$\big\|x_ju\nabla(-\Delta)^{-1}(u - V_0 - V_1 - J)\big\|_p \leqslant \|x_ju\|_r\big\|\nabla(-\Delta)^{-1}(u - V_0 - V_1 - J)\big\|_q$$
$$\leqslant C(1+t)^{-\gamma-\frac{3}{2}}\log(1+t). \tag{34}$$

Next, we derive the estimate for the second term on the right-hand side of (33). For this purpose, we prove the following inequality:

$$\big|\nabla(-\Delta)^{-1}V_0(1+t,x)\big| \leqslant C(1+t)^{-1}\big\langle(1+t)^{-1/2}x\big\rangle^{-2}. \tag{35}$$

Indeed, in Lemma 12, we put $n := 3$, $\varphi(\xi) := \mathcal{F}[\nabla(-\Delta)^{-1}V_0](1,\xi) = iM\xi|\xi|^{-2}e^{-|\xi|^2}$, $\lambda := -1$, $k := 1$ and $\mu := 1$. Then $|\nabla(-\Delta)^{-1}V_0(1,x)| \leqslant C|x|^{-2}$ is satisfied. Moreover, we obtain $|\nabla(-\Delta)^{-1}V_0(1,x)| = |\mathcal{F}^{-1}[\varphi](x)| \leqslant (2\pi)^{-3/2}\int_{\mathbb{R}^3}|\varphi(\xi)|d\xi \leqslant M(2\pi)^{-3/2}\int_{\mathbb{R}^3}|\xi|^{-1}e^{-|\xi|^2}\,d\xi \leqslant C$. Hence, we obtain the inequality $|\nabla(-\Delta)^{-1}V_0(1,x)| \leqslant C\langle x\rangle^{-2}$. A combination of this and the scaling property leads to the estimate (35). Thus, for $1 \leqslant q \leqslant \infty$ and $r > 3$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, we have

$$\begin{aligned}
&\big\|\big(u(t) - (V_0 + V_1 + J)(1+t)\big)x_j\nabla(-\Delta)^{-1}V_0(1+t)\big\|_p \\
&\quad\leqslant \big\|u(t) - (V_0 + V_1 + J)(1+t)\big\|_q\big\|x_j\nabla(-\Delta)^{-1}V_0(1+t)\big\|_r \\
&\quad\leqslant C(1+t)^{-\gamma-\frac{3}{2}}\log(1+t), \qquad (36)
\end{aligned}$$

where we use (29) and (35) in the second inequality. Employing a similar argument, the third term on the right-hand side of (33) satisfies

$$\big\|\big(u(t) - V_0(1+t)\big)x_j\nabla(-\Delta)^{-1}(V_1 + J)(1+t)\big\|_p \leqslant C(1+t)^{-\gamma-\frac{3}{2}}. \qquad (37)$$

Applying (34), (36) and (37) to (33), we conclude the desired estimate. □

Proof of Theorem 1. For the simplicity, we show only the case $\kappa = 1$. When $\kappa = -1$, we can prove this theorem by the similar way. By the integration by parts, we obtain the equality $\int_{\mathbb{R}^3} f(x)\nabla(-\Delta)^{-1}g(x)\,dx = -\int_{\mathbb{R}^3} g(x)\nabla(-\Delta)^{-1}f(x)\,dx$ for any $f, g \in L^{4/3}(\mathbb{R}^3)$. Hence, by (3), the solution u can be split into

$$\begin{aligned}
u(t) &= \sum_{2l+|\beta|\leqslant 2}\frac{\nabla^\beta\partial_t^l G(1+t,x)}{\beta!}(-1)^l\int_{\mathbb{R}^3}(-y)^\beta u_0(y)\,dy \\
&\quad - \sum_{|\beta|=1}\nabla^\beta\nabla G(1+t,x)\cdot\int_0^\infty\int_{\mathbb{R}^3} y^\beta\big\{u\nabla(-\Delta)^{-1}u(s,y) - \mathcal{V}(1+s,y)\big\}\,dy\,ds \\
&\quad + \int_0^t \nabla e^{(t-s)\Delta}\cdot\mathcal{V}(1+s,y)\,ds + \rho_0(t) + \rho_1(t) + \rho_2(t), \qquad (38)
\end{aligned}$$

where the function $\mathcal{V} = \mathcal{V}(t,x)$ is defined by (32), and we put

$$\begin{aligned}
\rho_0(t,x) &:= e^{t\Delta}u_0 - \sum_{2l+|\beta|\leqslant 2}\frac{(-1)^l\partial_t^l\nabla^\beta G(1+t,x)}{\beta}\int_{\mathbb{R}^3}(-y)^\beta u_0(y)\,dy, \\
\rho_1(t,x) &:= \int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big) \\
&\qquad\cdot\big(u\nabla(-\Delta)^{-1}u(s,y) - \mathcal{V}(1+s,y)\big)\,dy\,ds, \\
\rho_2(t,x) &:= \sum_{|\beta|=1}\nabla^\beta\nabla G(1+t,x)\cdot\int_t^\infty\int_{\mathbb{R}^3} y^\beta\big\{u\nabla(-\Delta)^{-1}u(s,y) - \mathcal{V}(1+s,y)\big\}\,dy\,ds.
\end{aligned}$$

Indeed, the nonlinear term on the right-hand side of (3) is split into

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(u\nabla(-\Delta)^{-1}u\big)(s)\,ds \\
&\quad = \int_0^t \nabla e^{(t-s)\Delta}\cdot\big(u\nabla(-\Delta)^{-1}u(s) - \mathcal{V}(1+s)\big)\,ds + \int_0^t \nabla e^{(t-s)\Delta}\cdot\mathcal{V}(1+s)\,ds. \qquad (39)
\end{aligned}$$

Applying the integration by parts, the integrand $u\nabla(-\Delta)^{-1}u - \mathcal{V}$ is vanishing. Hence, the first term on the right-hand side of (39) is split into

$$\int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u\nabla(-\Delta)^{-1}u(s) - \mathcal{V}(1+s)\big)\,ds$$
$$= \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^t \int_{\mathbb{R}^3} (-y)^\beta \big(u\nabla(-\Delta)^{-1}u(s,y) - \mathcal{V}(1+s,y)\big)\,dy\,ds + \rho_1(t)$$
$$= -\sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^\infty \int_{\mathbb{R}^3} y^\beta \big(u\nabla(-\Delta)^{-1}u(s,y) - \mathcal{V}(1+s,y)\big)\,dy\,ds + \rho_1(t) + \rho_2(t).$$

Substituting this into (39), we obtain (38). By the definition (32), the third term on the right-hand side of (38) is split into

$$\int_0^t \nabla e^{(t-s)\Delta} \cdot \mathcal{V}(1+s)\,ds = \int_0^t \nabla e^{(t-s)\Delta} \cdot V_0\nabla(-\Delta)^{-1}V_0(1+s)\,ds$$
$$+ \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(1+s)\,ds$$
$$+ \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)J + J\nabla(-\Delta)^{-1}V_0\big)(1+s)\,ds. \tag{40}$$

Since the odd integrand $V_0\nabla(-\Delta)^{-1}V_0$ is vanishing, we can split the first term on the right-hand side of (40) into

$$\int_0^t \nabla e^{(t-s)\Delta} \cdot V_0\nabla(-\Delta)^{-1}V_0(1+s)\,ds$$
$$= J(t) + \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_0(1+s) - V_0\nabla(-\Delta)^{-1}V_0(s)\big)\,ds$$
$$= J(1+t) - \big(J(1+t) - J(t)\big)$$
$$+ \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x)(-y)^\beta\Big)$$
$$\cdot \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds$$
$$+ \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^t \int_{\mathbb{R}^3} (-y)^\beta \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds$$
$$= \rho_3(t) + J(1+t)$$
$$- \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^\infty \int_{\mathbb{R}^3} y^\beta \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds, \tag{41}$$

where

$$\rho_3(t,x) := -\big(J(1+t) - J(t)\big)$$
$$+ \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x)(-y)^\beta\Big)$$
$$\cdot \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds$$
$$+ \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_t^\infty \int_{\mathbb{R}^3} y^\beta \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds.$$

Since the odd integrands $y_i(V_0\partial_j(-\Delta)^{-1}V_0)$ $(i \neq j)$ are vanishing, the third term on the right-hand side of (41) can be represented as

$$\begin{aligned}
&\sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^\infty \int_{\mathbb{R}^3} y^\beta \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds \\
&\quad = \sum_{j=1}^3 \partial_j^2 G(1+t,x) \int_0^\infty \int_{\mathbb{R}^3} y_j \big(V_0\partial_j(-\Delta)^{-1}V_0(1+s,y) - V_0\partial_j(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds \\
&\quad = \frac{1}{3}\Delta G(1+t,x) \int_0^\infty \int_{\mathbb{R}^3} y \cdot \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds,
\end{aligned}$$

where we use the relation $\int_{\mathbb{R}^3} y_j(V_0\partial_j(-\Delta)^{-1}V_0)\,dy = \frac{1}{3}\int_{\mathbb{R}^3} y \cdot (V_0\nabla(-\Delta)^{-1}V_0)\,dy$ $(1 \leqslant j \leqslant 3)$ in the second equality. Substituting this into the right-hand side of (41), we have

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta} \cdot V_0\nabla(-\Delta)^{-1}V_0(1+s)\,ds \\
&\quad = \rho_3(t) + J(1+t) - \frac{1}{3}\Delta G(1+t,x) \int_0^\infty \int_{\mathbb{R}^3} y \cdot \big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y) - V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds. \qquad (42)
\end{aligned}$$

Similarly, since the integrands $y^\beta(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0)$ $(|\beta| \leqslant 1)$ are vanishing, the second term on the right-hand side of (40) is represented as

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(1+s)\,ds \\
&\quad = \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds \\
&\qquad + \int_0^t \nabla e^{(t-s)\Delta} \cdot \Big(\big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(1+s) - \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\Big)\,ds \\
&\quad = \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds + \rho_4(t,x), \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
\rho_4(t,x) := &\Bigg\{ \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds \\
&- \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds \Bigg\} \\
&+ \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x)(-y)^\beta \Big) \\
&\cdot \Big\{ \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(1+s,y) - \big(V_0\nabla(-\Delta)^{-1}V_1 + V_1\nabla(-\Delta)^{-1}V_0\big)(s,y) \Big\}\,dy\,ds.
\end{aligned}$$

Next, we consider the third term on the right-hand side of (40). Since the integrands $V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0$ and $y_j(V_0\partial_k(-\Delta)^{-1}J + J\partial_k(-\Delta)^{-1}V_0)$ $(j \neq k)$ are vanishing, the third term on the right-hand side of (40) is split into

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s)\,ds\\
&\quad=\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(t,x)\cdot\int_0^t\int_{\mathbb{R}^3}(-y)^\beta\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds\\
&\qquad+\int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(t,x)(-y)^\beta\Big)\\
&\qquad\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds\\
&\quad=-\frac{1}{3}\Delta G(t,x)\int_0^t\int_{\mathbb{R}^3}y\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds\\
&\qquad+\int_0^{1+t}\int_{\mathbb{R}^3}\Big(\nabla G(1+t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big)\\
&\qquad\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds+\rho_5(t),
\end{aligned}\tag{44}$$

where we use the relation

$$\int_{\mathbb{R}^3}y_j\big(V_0\partial_j(-\Delta)^{-1}J+J\partial_j(-\Delta)^{-1}V_0\big)\,dy=\frac{1}{3}\int_{\mathbb{R}^3}y\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)\,dy$$

for $1\leqslant j\leqslant 3$ in the second equality, and we put

$$\begin{aligned}
\rho_5(t,x):=&\Bigg\{\int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(t,x)(-y)^\beta\Big)\\
&\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds\\
&-\int_0^{1+t}\int_{\mathbb{R}^3}\Big(\nabla G(1+t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big)\\
&\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds\Bigg\}\\
&+\int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(t,x)(-y)^\beta\Big)\\
&\cdot\big\{\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)-\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(s,y)\big\}\,dy\,ds.
\end{aligned}$$

By the scaling property (6), the first term on the right-hand side of (44) is split into

$$\frac{1}{3}\Delta G(t,x)\int_0^t\int_{\mathbb{R}^3}(-y)\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds=\rho_6(t,x)+K(1+t,x),$$

where the function $K=K(t,x)$ is defined by (8), and $\rho_6=\rho_6(t,x)$ is provided by

$$\begin{aligned}
\rho_6(t,x):=&\frac{1}{3}\Delta G(t,x)\int_0^t\int_{\mathbb{R}^3}(-y)\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds\\
&-\frac{1}{3}\Delta G(1+t,x)\int_0^{1+t}\int_{\mathbb{R}^3}(-y)\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(1+s,y)\,dy\,ds.
\end{aligned}$$

Thus, the third term on the right-hand side of (40) is given by

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)(1+s)\,ds\\
&\quad=\rho_5(t,x)+\rho_6(t,x)+K(1+t,x)\\
&\qquad+\int_0^{1+t}\int_{\mathbb{R}^3}\Big(\nabla G(1+t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big)\\
&\qquad\cdot\big(V_0\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds. \qquad (45)
\end{aligned}$$

Substituting (42), (43) and (45) into (40), the third term on the right-hand side of (38) can be represented as follows:

$$\begin{aligned}
\int_0^t \nabla e^{(t-s)\Delta}\cdot\mathcal{V}(1+s)\,ds &= \rho_3(t)+\rho_4(t)+\rho_5(t)+\rho_6(t)+J(1+t)+K(1+t)\\
&\quad+\int_0^{1+t}\nabla e^{(1+t-s)\Delta}\cdot\big(V_0\nabla(-\Delta)^{-1}V_1+V_1\nabla(-\Delta)^{-1}V_0\big)(s)\,ds\\
&\quad+\int_0^{1+t}\int_{\mathbb{R}^3}\Big(\nabla G(1+t-s,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big)\\
&\quad\cdot\big(V_0\nabla(-\Delta)^{-1}J+J\nabla(-\Delta)^{-1}V_0\big)(s,y)\,dy\,ds\\
&\quad-\frac{1}{3}\Delta G(1+t,x)\int_0^\infty\int_{\mathbb{R}^3} y\cdot\big(V_0\nabla(-\Delta)^{-1}V_0(1+s,y)-V_0\nabla(-\Delta)^{-1}V_0(s,y)\big)\,dy\,ds.
\end{aligned}$$

Substituting this into (38), we obtain

$$u(t)=V_0(1+t)+V_1(1+t)+J(1+t)+K(1+t)+V_2(1+t)+\rho_0(t)+\cdots+\rho_6(t). \qquad (46)$$

We should confirm that the functions $\{\rho_j(t)\}_{j=0}^6$ satisfy

$$\big\|\rho_j(t)\big\|_p=o\big(t^{-\gamma-1}\big)\quad\text{as } t\to\infty \text{ for } 1\leqslant p\leqslant\infty,\ \gamma:=\frac{3}{2}\Big(1-\frac{1}{p}\Big). \qquad (47)$$

It is well known that ρ_0 satisfies (47). In order to obtain the estimate (47) for ρ_1 in (46), we introduce a positive function $R=R(t)$ with $R(t)\to\infty$, $R(t)=o(t^{1/2})$ as $t\to\infty$. By employing the function $R(t)$ and the mean-valued theorem, the function ρ_1 is split into

$$\begin{aligned}
\rho_1(t)&=\int_0^t\int_{\mathbb{R}^3}\big(\nabla G(t-s,x-y)-\nabla G(1+t,x-y)\big)\cdot\big(u\nabla(-\Delta)^{-1}u(s,y)-\mathcal{V}(1+s,y)\big)\,dy\,ds\\
&\quad+\int_0^t\Big(\int_{|y|\leqslant R(t)}+\int_{|y|\geqslant R(t)}\Big)\Big(\nabla G(1+t,x-y)-\sum_{|\beta|\leqslant 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big)\\
&\quad\cdot\big(u\nabla(-\Delta)^{-1}u(s,y)-\mathcal{V}(1+s,y)\big)\,dy\,ds\\
&=\rho_{1,1}(t)+\rho_{1,2}(t)+\rho_{1,3}(t),
\end{aligned}$$

where the function $\mathcal{V}=\mathcal{V}(t,x)$ is defined by (32), and we put

$$\rho_{1,1}(t):=\int_0^t\int_{\mathbb{R}^3}\int_0^1\partial_t\nabla G\big(t-s+\sigma(1+s),x-y\big)\,d\sigma\cdot(-1-s)\big(u\nabla(-\Delta)^{-1}u(s,y)-\mathcal{V}(1+s,y)\big)\,dy\,ds,$$

$$\rho_{1,2}(t):=\sum_{|\beta|=2}\frac{1}{\beta!}\int_0^t\int_{|y|\leqslant R(t)}\int_0^1(1-\sigma)\nabla^\beta\nabla G(1+t,x-\sigma y)\,d\sigma\cdot(-y)^\beta\big(u\nabla(-\Delta)^{-1}u(s,y)-\mathcal{V}(1+s,y)\big)\,dy\,ds,$$

$$\rho_{1,3}(t) := \sum_{|\beta|=1} \int_0^t \int_{|y|\geqslant R(t)} \left(\int_0^1 \nabla^\beta \nabla G(1+t, x-\sigma y)\, d\sigma - \nabla^\beta \nabla G(1+t, x) \right)$$
$$\cdot (-y)^\beta \big(u\nabla(-\Delta)^{-1} u(s,y) - \mathcal{V}(1+s,y) \big)\, dy\, ds.$$

Applying Hausdorff–Young's inequality and Proposition 11, we have

$$\big\| \rho_{1,1}(t) \big\|_p \leqslant C(1+t)^{-\gamma-\frac{3}{2}} \log(2+t), \qquad \big\| \rho_{1,2}(t) \big\|_p \leqslant C(1+t)^{-\gamma-\frac{3}{2}} \log(2+t) R(t).$$

Since $R(t) = o(t^{1/2})$ as $t \to \infty$, we obtain $\|\rho_{1,1}(t)\|_p, \|\rho_{1,2}(t)\|_p = o(t^{-\gamma-1})$ as $t \to \infty$ for $1 \leqslant p \leqslant \infty$, $\gamma := \frac{3}{2}(1 - \frac{1}{p})$. On the other hand, the function $\rho_{1,3}(t)$ satisfies

$$\big\| \rho_{1,3}(t) \big\|_p \leqslant \sum_{|\beta|=1} \big\| \nabla^\beta \nabla G(1+t) \big\|_p \int_0^t \int_{|y|\geqslant R(t)} \big| y^\beta \big\{ u\nabla(-\Delta)^{-1} u(s,y) - \mathcal{V}(1+s,y) \big\} \big|\, dy\, ds.$$

Proposition 11 and the condition of $R(t)$ give

$$\int_0^t \int_{|y|\geqslant R(t)} \big| y^\beta \big\{ u\nabla(-\Delta)^{-1} u(s,y) - \mathcal{V}(1+s,y) \big\} \big|\, dy\, ds = o(1) \quad \text{as } t \to \infty.$$

Hence, the function $\rho_{1,3}$ satisfies $\|\rho_{1,3}(t)\|_p = o(t^{-\gamma-1})$ as $t \to \infty$ for $1 \leqslant p \leqslant \infty$ and $\gamma := \frac{3}{2}(1 - \frac{1}{p})$. Thus, the function ρ_1 in (46) satisfies the estimate (47). Proposition 11 gives the estimate (47) for ρ_2. The mean-valued theorem and Proposition 5 give that the functions ρ_3, ρ_4, ρ_5 and ρ_6 satisfy the estimate (47). Now, we obtain the estimate (47) for all $0 \leqslant j \leqslant 6$. Thus, we conclude the proof. □

4. Proof of Theorem 2

In this section, we prove Theorem 2. We can split the right-hand side of (3) into

$$\begin{aligned} u(t,x) &= V_0(1+t,x) + V_1(1+t,x) + \sum_{2l+|\beta|=2} \frac{(-1)^l \partial_t^l \nabla^\beta G(1+t,x)}{\beta!} \int_{\mathbb{R}^4} (-y)^\beta u_0(y)\, dy \\ &\quad - \frac{\kappa}{4} \Delta G(t,x) \int_0^t \int_{\mathbb{R}^4} y \cdot V_0 \nabla(-\Delta)^{-1} V_0(1+s,y)\, dy\, ds + \tilde{J}(t,x) \\ &\quad - \kappa \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_0^\infty \int_{\mathbb{R}^4} y^\beta \big(u\nabla(-\Delta)^{-1} u(s,y) - V_0 \nabla(-\Delta)^{-1} V_0(1+s,y) \big)\, dy\, ds \\ &\quad + \varrho_0(t,x) + \varrho_1(t,x) + \varrho_2(t,x) + \varrho_3(t,x), \end{aligned} \tag{48}$$

where the functions $V_0 = V_0(t,x)$ and $V_1 = V_1(t,x)$ are defined by (4), and we put

$$\varrho_0(t,x) := e^{t\Delta} u_0 - \sum_{2l+|\beta|\leqslant 2} \frac{(-1)^l \partial_t^l \nabla^\beta G(1+t,x)}{\beta!} \int_{\mathbb{R}^4} (-y)^\beta u_0(y)\, dy,$$

$$\varrho_1(t,x) := \kappa \int_0^t \int_{\mathbb{R}^4} \Big(\nabla G(t-s, x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \Big)$$
$$\cdot \big(V_0 \nabla(-\Delta)^{-1} V_0(1+s,y) - V_0 \nabla(-\Delta)^{-1} V_0(s,y) \big)\, dy\, ds,$$

$$\varrho_2(t,x) := \kappa \int_t^\infty \int_{\mathbb{R}^4} (y \cdot \nabla) \nabla G(1+t,x) \cdot \big(u\nabla(-\Delta)^{-1} u(s,y) - V_0 \nabla(-\Delta)^{-1} V_0(1+s,y) \big)\, dy\, ds,$$

$$\varrho_3(t,x) := \kappa \int_0^t \int_{\mathbb{R}^4} \Big(\nabla G(t-s, x-y) - \sum_{|\beta|\leqslant 1} \nabla^\beta \nabla G(1+t,x)(-y)^\beta \Big)$$
$$\cdot \big(u\nabla(-\Delta)^{-1} u(s,y) - V_0 \nabla(-\Delta)^{-1} V_0(1+s,y) \big)\, dy\, ds,$$

$$\tilde{J}(t,x) := \kappa \int_0^t \int_{\mathbb{R}^4} \biggl(\nabla G(t-s, x-y) - \sum_{|\beta| \leqslant 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \biggr) \cdot \bigl(V_0 \nabla (-\Delta)^{-1} V_0 \bigr)(s,y)\, dy\, ds.$$

Indeed, since the integrands $u\nabla(-\Delta)^{-1}u$, $V_0\nabla(-\Delta)^{-1}V_0$ and $y_j V_0 \partial_k (-\Delta)^{-1} V_0$ for $j \neq k$ are vanishing, the nonlinear term on the right-hand side of (3) is split into

$$\begin{aligned}
&\kappa \int_0^t \nabla e^{(t-s)\Delta} \cdot \bigl(u \nabla (-\Delta)^{-1} u \bigr)(s)\, ds \\
&\quad = \kappa \int_0^t \nabla e^{(t-s)\Delta} \cdot \bigl(V_0 \nabla (-\Delta)^{-1} V_0 \bigr)(1+s)\, ds \\
&\qquad + \kappa \int_0^t \nabla e^{(t-s)\Delta} \cdot \bigl(u \nabla (-\Delta)^{-1} u(s) - V_0 \nabla (-\Delta)^{-1} V_0 (1+s) \bigr)\, ds \\
&\quad = \kappa \int_0^t \int_{\mathbb{R}^4} \biggl(\nabla G(t-s, x-y) - \sum_{|\beta| \leqslant 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \biggr) \cdot \bigl(V_0 \nabla (-\Delta)^{-1} V_0 \bigr)(1+s, y)\, dy\, ds \\
&\qquad + \kappa \sum_{j=1}^4 \partial_j^2 G(t,x) \int_0^t \int_{\mathbb{R}^4} (-y_j) \bigl(V_0 \partial_j (-\Delta)^{-1} V_0 \bigr)(1+s,y)\, dy\, ds + \varrho_2(t,x) + \varrho_3(t,x) \\
&\qquad + \kappa \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t, x) \cdot \int_0^\infty \int_{\mathbb{R}^4} (-y)^\beta \bigl(u \nabla (-\Delta)^{-1} u(s,y) - V_0 \nabla (-\Delta)^{-1} V_0 (1+s,y) \bigr)\, dy\, ds \\
&\quad = \tilde{J}(t,x) + \varrho_1(t,x) - \frac{\kappa}{4} \Delta G(t,x) \int_0^t \int_{\mathbb{R}^4} y \cdot V_0 \nabla (-\Delta)^{-1} V_0 (1+s,y)\, dy\, ds + \varrho_2(t,x) + \varrho_3(t,x) \\
&\qquad + \kappa \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t, x) \cdot \int_0^\infty \int_{\mathbb{R}^4} (-y)^\beta \bigl(u \nabla (-\Delta)^{-1} u(s,y) - V_0 \nabla (-\Delta)^{-1} V_0 (1+s,y) \bigr)\, dy\, ds,
\end{aligned}$$

where we use the relation $\int_{\mathbb{R}^4} y_j (V_0 \partial_j (-\Delta)^{-1} V_0)(1+s,y)\, dy = \frac{1}{4} \int_{\mathbb{R}^4} y \cdot (V_0 \nabla (-\Delta)^{-1} V_0)(1+s,y)\, dy$ $(1 \leqslant j \leqslant 4)$ in the third equality. Hence, we obtain (48). Thus, the solution u is split into

$$u(t) = V_0(1+t) + V_1(1+t) + \tilde{K}(1+t) + \tilde{V}_2(1+t) + \varrho_0(t) + \cdots + \varrho_5(t),$$

where the functions $\tilde{K} = \tilde{K}(t,x)$ and $\tilde{V}_2 = \tilde{V}_2(t,x)$ are defined by (11), and we put

$$\varrho_4(t,x) := \tilde{J}(t,x) - \tilde{J}(1+t,x), \qquad \varrho_5(t,x) := \tilde{K}(t,x) - \tilde{K}(1+t,x).$$

We should confirm that $\{\varrho_j\}_{j=0}^5$ satisfy

$$\bigl\| \varrho_j(t) \bigr\|_p = o\bigl(t^{-\gamma-1} \bigr) \quad \text{as } t \to \infty \text{ for } 1 \leqslant p \leqslant \infty,\ \gamma := 2\biggl(1 - \frac{1}{p} \biggr). \tag{49}$$

It is well known that ϱ_0 satisfies (49). By the mean-valued theorem and Proposition 8, we can obtain the estimate (49) for $1 \leqslant j \leqslant 5$. Thus, we conclude the proof.

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Appendix A. Higher-dimensional cases

Theorems 1 and 2 state that the asymptotic profile of the solution in $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ contains no logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k}{2}}$ $(0 \leqslant k \leqslant 1)$ when the dimension $n = 3$ or 4. Generally speaking, when $n \geqslant 4$ is even, we can prove that a logarithmic term does not appear in the asymptotic profile of the solution at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k}{2}}$ for $0 \leqslant k \leqslant n-3$. On the other hand, when $n \geqslant 3$ is odd, a logarithmic term never occurs in the asymptotic expansion of the solution at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k}{2}}$ for $0 \leqslant k \leqslant 2n-5$. Moreover, Theorems 1 and 2 state that the logarithmic term is included in the asymptotic profile of the solution at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-1}$ when $n = 3$ or 4. By the similar arguments as in the proof of those theorems, we can expect that the asymptotic expansion of the solution contains a logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-2}{2}}$ when $n \geqslant 6$ is even; and at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-(n-2)}$ when $n \geqslant 5$ is odd. More precisely, when $n \geqslant 6$ is even, the asymptotic expansion of the solution contains the following term:

$$\begin{aligned}
\tilde{K}_n(t,x) &:= \sum_{2l+|\beta|=n-3} \frac{\partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \cdot \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta \big(V_0 \nabla(-\Delta)^{-1} V_0\big)(1+s,y)\,dy\,ds \\
&= \sum_{2l+|\beta|=n-3} \frac{(-1)^l \partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \int_0^t (1+s)^{-1}\,ds \\
&\quad \cdot \int_{\mathbb{R}^n} \big(-(1+s)^{-1/2} y\big)^\beta \big(V_0 \nabla(-\Delta)^{-1} V_0\big)\big(1,(1+s)^{-1/2} y\big)(1+s)^{-n/2}\,dy \\
&= \sum_{2l+|\beta|=n-3} \frac{(-1)^l \partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \log(1+t) \cdot \int_{\mathbb{R}^n} (-\eta)^\beta \big(V_0 \nabla(-\Delta)^{-1} V_0\big)(1,\eta)\,d\eta,
\end{aligned}$$

where the function V_0 is defined by (4), we use the relation (6) in the second equality, and we put $\eta := (1+s)^{-1/2} y$ in the third equality. When $n = 4$, this term provides $\tilde{K}$ in Theorem 2. If the coefficient satisfies $\sum_{2l+|\beta|=n-3} \frac{(-1)^l \partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \cdot \int_{\mathbb{R}^n} (-\eta)^\beta (V_0 \nabla(-\Delta)^{-1} V_0)(1,\eta)\,d\eta \not\equiv 0$, then we can conclude that the asymptotic expansion of the solution contains the logarithmic term at the rate $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-2}{2}}$. On the other hand, when $n \geqslant 5$ is odd, the logarithmic term in the asymptotic expansion of the solution is given by the following function:

$$\begin{aligned}
K_n(t,x) &:= \sum_{2l+|\beta|=2n-5} \frac{\partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \cdot \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta \big(V_0 \nabla(-\Delta)^{-1} J_n + J_n \nabla(-\Delta)^{-1} V_0\big)(1+s,y)\,dy\,ds \\
&= \sum_{2l+|\beta|=2n-5} \frac{(-1)^l \partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \int_0^t (1+s)^{-1}\,ds \\
&\quad \cdot \int_{\mathbb{R}^n} \big(-(1+s)^{-1/2} y\big)^\beta \big(V_0 \nabla(-\Delta)^{-1} J_n + J_n \nabla(-\Delta)^{-1} V_0\big)\big(1,(1+s)^{-1/2} y\big)(1+s)^{-n/2}\,dy \\
&= \sum_{2l+|\beta|=2n-5} \frac{(-1)^l \partial_t^l \nabla^\beta \nabla G(1+t,x)}{l!\beta!} \log(1+t) \cdot \int_{\mathbb{R}^n} (-\eta)^\beta \big(V_0 \nabla(-\Delta)^{-1} J_n + J_n \nabla(-\Delta)^{-1} V_0\big)(1,\eta)\,d\eta,
\end{aligned}$$

where we define the function $J_n = J_n(t,x)$ by

$$J_n(t,x) := \int_0^t \int_{\mathbb{R}^n} \Bigg(\nabla G(t-s,x-y) - \sum_{2l+|\beta|\leqslant n-4} \frac{\partial_t^l \nabla^\beta \nabla G(t,x)}{l!\beta!} (-s)^l (-y)^\beta\Bigg) \cdot V_0 \nabla(-\Delta)^{-1} V_0(s,y)\,dy\,ds,$$

we use the relation $\lambda^{2n-2} J_n(\lambda^2 t, \lambda x) = J_n(t,x)$ $(\lambda > 0)$ in the second equality, and we put $\eta := (1+s)^{-1/2} y$ in the third equality. The functions K_n and J_n are extension of the functions K and J in Theorem 1, respectively. We can expect that $K_n \not\equiv 0$ is satisfied. Indeed, since the function J_n is even in η, then the integrand $(-\eta)^\beta (V_0 \nabla(-\Delta)^{-1} J_n + J_n \nabla(-\Delta)^{-1} V_0)(1,\eta)$ is even for arbitrary $\beta \in \mathbb{Z}_+^n$ with $|\beta| = 2n-2l-5$. If we prove $K_n \not\equiv 0$, then we can obtain the desired conclusion for multi-dimensional cases.

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